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1991 J. Phys. A: Math. Gen. 24 L197

(http://iopscience.iop.org/0305-4470/24/4/007)

J. Phys. A: Math. Gen. 24 (1991) L197–L200. Printed in the UK

LETTER TO THE EDITOR

Critical exponents from the large N expansion for the three-dimensional O(3) σ model

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Received 19 November 1990

Abstract. Using resummation techniques, we extract estimates for the critical exponents $1/\nu$ and η of the three-dimensional O(3) σ model from the known $O(1/N^2)$ and $O(1/N^3)$ field theory corrections, respectively, and compare the former with recent Monte Carlo results.

The three-dimensional O(3) σ model has been of interest lately due to its close connection with models which describe high T_c superconductors [1]. For instance, the Hubbard model is related, in a particular limit, to the Heisenberg model [2], which can be described at the critical point of the theory by the O(3) σ model, which is a continuum field theory [3]. Subsequently, various authors [4, 5] have studied the three-dimensional σ model at finite temperature using Monte Carlo simulations on large lattices. They have, in particular, obtained numerical estimates for the critical exponent $1/\nu$ which is related to the β-function of the renormalization group equation at criticality via $1/\nu = -\beta'(g_c)$, where g_c is the (dimensionless) critical coupling in d dimensions. The estimate was obtained by examining the model in the zero temperature limit, where an order–disorder phase transition occurs, and the slope of the β-function was determined from a linear fit to the data.

The three-dimensional model has also been examined analytically using perturbation theory in [6] and the (non-perturbative) large N approach [7]. In [6], a three loop perturbative renormalization was performed in minimal subtraction near two dimensions and the d-dimensional β-function was then determined, from which a numerical estimate was obtained for $1/\nu$ in $d=3$. The non-trivial zero of this β-function gives the location of the phase transition and corresponds to the zero temperature limit of the lattice model. First, the exponent ν was expressed as a double power series in $\varepsilon = d-2$ and $1/(N-2)$ valid to the three loop approximation. Subsequently setting $N=3$, one obtains a divergent series which was resummed by a Borel transformation, whose integrand was then replaced by a [1, 2] Padé approximant. Substituting $\varepsilon=1$ in the final expression yielded $1/\nu = 1.25$ [6], which is within the error bounds of the recent Monte Carlo result [4, 5], which gave $1/\nu = 1.28 \pm 0.05$. Unfortunately, to improve on this estimate by including the four loop term of the two-dimensional β-function, which has been determined in [8, 9], one encounters a Borel singularity which obstructs any resummation. (A simple pole also appears when one uses the two loop result [6].) One would instead have to use the five loop result which is currently unavailable as it appears that the Borel singularity which occurs when considering an even order of perturbation theory moves off the axis of integration for an odd order.

0305-4470/91/040197+04$03.50

In the alternative non-perturbative approach to dealing with the O(3) σ model, one replaces the symmetry group by $O(N)$ and uses $1/N$ as the expansion parameter where N is large, eventually setting $N=3$ at the end of the calculation. As the model is renormalizable in three dimensions in this approach [7], one can calculate the large N approximation to the critical exponents. This has been carried out to $O(1/N^2)$ for $1/\nu$ in three dimensions in [10] and in arbitrary dimensions in [11, 12], and at $O(1/N^3)$ for η in [13], where the previous few orders were determined in [11, 14, 15]. However, the numerical estimates one obtains for the exponents by replacing N by 3 at the end of the calculation are not in close agreement with results of other methods [16], and indeed appear to diverge. Thus it is the purpose of this letter to apply resummation techniques, similar to those of the approach of [6], to obtain improved numerical estimates from known *analytic* expressions of the exponents and to demonstrate that one can obtain an estimate for $1/\nu$ in close agreement with lattice calculations. Indeed we believe this large N resummation has not been carried out before.

We begin by noting the values of the various exponents for the O(N) σ model in three dimensions to the orders they are known. First [11, 13–15]

$$\eta=\frac{\eta_1}{N}-\frac{8}{3}\left(\frac{\eta_1}{N}\right)^2+\left[\frac{9}{2}\pi^2\ln 2+\frac{27}{8}\psi''\left(\frac{1}{2}\right)-\frac{61}{24}\pi^2-\frac{797}{18}\right]\left(\frac{\eta_1}{N}\right)^3 \tag{1}$$

where $\eta_1=8/(3\pi^2)$ and $\psi(x)$ is the logarithmic derivative of the Γ-function. Also [11, 12]

$$\nu=1-\frac{4\eta_1}{N}+\left[\tfrac{56}{3}-\tfrac{9}{2}\pi^2\right]\left(\frac{\eta_1}{N}\right)^2 \tag{2}$$

where the expansion parameter is $1/N$. It was observed in [13] that setting $N=3$ in (1), one obtains a value for η which is beginning to diverge from the accepted value of 0.04, deduced by ε-expansions [16, 17]. This perhaps indicates that (1) is asymptotic with respect to the parameter N. However, Borel summing the three terms of (1) we obtain

$$\eta=N\eta_1\int_0^\infty \mathrm{d}t\, t\, \mathrm{e}^{-Nt}\left[1-\frac{4}{3}\eta_1 t+\frac{\eta_1^2t^2}{4}\left(3\pi^2\ln 2-\frac{63}{2}\zeta(3)-\frac{61\pi^2}{36}-\frac{797}{27}\right)\right] \tag{3}$$

where $\zeta(n)$ is the Riemann zeta function. Replacing the Borel transform by a [1, 2] Padé approximant and setting $N=3$, leads to

$$\eta=\frac{8}{\pi^2}\int_0^\infty \mathrm{d}t\, t\, \mathrm{e}^{-3t}\left\{1+\frac{32t}{9\pi^2}+\left(\frac{4t}{3\pi^2}\right)^2\left[\frac{989}{27}+\frac{61\pi^2}{36}+\frac{63\zeta(3)}{2}-3\pi^2\ln 2\right]\right\}^{-1}$$
$$=0.05 \tag{4}$$

for the O(3) model, which at least gives the correct order of magnitude, and is an improved estimate on the three loop result of [9] i.e. 0.11. Moreover, if only the first two terms of (1) are included, then one obtains an estimate of 0.07 for η, which suggests that the resummation we employ is converging to the accepted value.

Thus having illustrated that the Padé–Borel approach is successful for η, we can apply it equally to $1/\nu$. First, from (2), we again note that setting $N=3$, the exponent is diverging from the Monte Carlo result [4, 5], since the subsequent corrections would give $1/\nu = 1.70$, whilst the first two terms of (2) give $1/\nu = 1.36$. However, resumming the series in a similar way to η, we have

$$\nu = 3\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-3t}\left\{1+\frac{16t}{3\pi^2}+\left(\frac{8t}{3\pi^2}\right)^2\left[\frac{8}{9}+\frac{3\pi^2}{4}\right]\right\}^{-1}$$

$$= 0.813 \tag{5}$$

which gives $1/\nu = 1.23$ for the O(3) model, which is within the error bounds of the MC computation on the lattice. Moreover, considering the first two terms of (2) only, one obtains $1/\nu = 1.19$, which implies that the $1/N$ expansion is again also converging towards the Monte Carlo value of 1.28 ± 0.05. We have summed ν here and then calculated its inverse, rather than $1/\nu$ as in summing the latter one encounters Borel singularities, i.e. poles in the right half Borel plane, which therefore obstruct the summation.

We conclude with various remarks. First, in the resummation of the three-dimensional large N exponents no Borel singularities were encountered, in contrast to the situation when an even number of orders in perturbation theory are considered in the double expansion in $(d-2)$ and $1/(N-2)$ of the perturbative approach [6]. This is due to the fact that in the latter case one is using a result which is valid near two dimensions, where $\varepsilon \ll 1$, and extending it to a region where $\varepsilon = 1$ where the model becomes perturbatively non-renormalizable. It is not clear therefore whether the two-dimensional result contains all the information required for extending to this region. Terms such as $\exp(-1/\varepsilon)$, which are non-analytic in the coupling constant in two dimensions, i.e. non-perturbative, would be significant in three dimensions, but their presence in the two-dimensional β-function used in [6] can never be calculated from perturbation theory. As suggested by Brézin [6], one would presumably have to include terms such as $\exp(-c/\varepsilon)$ in the perturbative β-function to overcome the Borel singularities. By contrast, the large N expansion of the model is renormalizable in three dimensions, as well as two, and is non-perturbative, being a reordering of perturbation theory such that chains of bubbles are summed first. Thus, no non-analytic terms are omitted. Further, the exponents calculated in this expansion have been determined in arbitrary dimensions in [12, 13, 15] and so there is therefore no extension to a region where the validity of the expansion is in question when one considers three dimensions.

Secondly, we have not used other Padé approximants, such as [2, 1], for either large N exponents for two reasons. First, in general terms such an approximant would in some sense be unnatural. For instance, the [2, 1] Padé approximant will involve the first two terms of the large N expansion of their exponents in the numerator and the third term in the denominator of the integrand of the Borel transform. Thus if one were to consider the [2, 1] approximant, its natural predecessor, in the sequence of approximations used to understand the convergence, is a [2, 0] approximant, i.e. just the Borel transform itself. In practice though, with (1) and (2) it turns out both [2, 1] Padé approximants have Borel singularities and cannot therefore be summed.

Finally, we have compared our results for the exponent η with the ε-expansion of the ϕ^4 theory whilst the exponent $1/\nu$ with the Monte Carlo treatment of the O(3) σ model, which may appear inconsistent. However, both models are equivalent at the critical point and therefore the exponents corresponding to the anomalous dimensions of each field ought to be equivalent. The ε-expansion result for η is known very accurately [16, 17] and it is encouraging that our large N treatment of η is converging towards this result. Similarly, $1/\nu$ is also known accurately from the ε-expansion, but we have compared the result we obtain for it with other calculations in the O(3) σ model, such as the Monte Carlo result [4, 5], and obtain good agreement with it, though the ε-expansion result, derived from the ϕ^4 model, gives a value of 1.41 [17]. The reason for this discrepancy can perhaps be understood from the method used to deduce the $O(1/N)$ and higher terms of (1) and (2). For instance, in [12, 13, 15] the asymptotic scaling part of the propagators are solved for in the form $A[1+A'(x^2)^{1/\nu}]/(x^2)^\alpha$, where A and A' are amplitudes and α the exponent of the field and which depends on η. The exponent $1/\nu$ is solved by considering the corrections to the leading order term of this propagator. However, it is not clear whether the equivalence of the ϕ^4 model and the O(3) σ model is preserved beyond the above leading order. Indeed our result for $1/\nu$, (5), is closer in agreement with the result of [4, 5] as well as the three loop perturbative result of [6] and a more recent calculation of Pis'mak and Polyakov [18] which involved solving the self-consistency equations for critical exponents and gave $1/\nu = 1.19$ for the O(3) σ model. (They obtained $\eta = 0.04$ by the same method [18].) As these methods are all different, but give very similar results for $1/\nu$, this would appear to substantiate our point of view that the discrepancy could be perhaps due to the inequivalence of both models beyond a certain point.

The author thanks Drs J R Honkonen and Yu M Pis'mak for discussions and for providing reference [18].

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